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Construction and Classification of Minimal Representations of Semi-separable Kernels

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An algorithm to construct a minimal lower separable representation out of a lower separable representation of a kernel of an integral operator is given. The minimal lower separable representations of a kernel are classified. The number of non-similar minimal lower separable representations of a kernel is discussed.
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0. INTRODUCTION AND PRELIMINARIES

Recently, in [6], the simplest separable representations of kernels k of integral operators on upper and lower triangular parts of a square $[a, b] \times [a, b]$ in $\mathbb{R}^2$ were analysed. The discussed operators were

$$K: L_2([a, b], Z) \rightarrow L_2([a, b], Y),$$

defined by

$$(K\varphi)(t) = \int_a^b k(t, s)\, \varphi(s)\, ds,$$

where Z and Y are finite dimensional inner product spaces and $L_2([a, b], U)$ denotes the space of L_2 functions on $[a, b]$ to U, for a finite dimensional inner product space U. The pair $\{F, G\}$ is called a *lower (upper) separable representation* of the kernel k, when $F(t): X \rightarrow Y$ and $G(t): Z \rightarrow X$ for t in $[a, b]$, where X is a finite dimensional inner product space; the entries of the matrices of F and G are square integrable and finally:

$$k(t, s) = F(t)\, G(s) \qquad \text{for } a \leqslant s < t \leqslant b \ (a \leqslant t < s \leqslant b), \quad \text{a.e.}$$

The space X is called the internal space and its dimension is the order of $\{F, G\}$. If k has a lower (upper) separable representation, then we call k

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lower (*upper*) *separable*. A lower (upper) separable representation $\{F, G\}$ of k is *minimal*, when the representation $\{F, G\}$ has the smallest possible order among all lower (upper) separable representations. The order of a minimal lower (upper) separable representation is the *lower* (*upper*) *order* of k, notation $l(k)$ $(u(k))$. The condition that the operators,

$$\int_\gamma^b F(t)^* F(t)\, dt,$$

$$\int_a^\gamma G(s)\, G(s)^*\, dt, \tag{0.1}$$

are invertible for some γ in (a, b), is sufficient for $\{F, G\}$ to be a minimal lower separable representation. To classify the various separable representations the concepts of similarity and reduction are used. Two separable representations of k, $\{F_1, G_1\}$ and $\{F_2, G_2\}$ with internal spaces X_1 and X_2, respectively, are *similar* if there exist an invertible operator $S\colon X_1 \to X_2$, such that

$$F_1(t) = F_2(t)\, S, \qquad a \leqslant t \leqslant b, \text{ a.e.}$$

$$G_1(s) = S^{-1} G_2(s), \qquad a \leqslant s \leqslant b, \text{ a.e.}$$

The pair $\{F_0, G_0\}$, with internal space X_0, is called a *reduction* of $\{F, G\}$ with internal space X and $\{F, G\}$ a *dilation* of $\{F_0, G_0\}$, if there exist spaces $X_1, X_2 \subset X$ such that

$$X = X_1 \oplus X_0 \oplus X_2$$

and relative to this decomposition $F(t)$ and $G(s)$ have the following form:

$$F(t) = [0 \quad F_0(t) \quad *], \qquad a \leqslant t \leqslant b, \text{ a.e.}$$

$$G(s) = \begin{bmatrix} * \\ G_0(s) \\ 0 \end{bmatrix}, \qquad a \leqslant s \leqslant b, \text{ a.e.}$$

The pair $\{F, G\}$ is said to be *irreducible* when it has no proper reduction; i.e., $\{F, G\}$ has no reduction $\{F_0, G_0\}$ such that the order of $\{F_0, G_0\}$ is strictly smaller than the order of $\{F, G\}$.

When the kernel k has a lower separable representation, it is possible to construct an irreducible one (see [6], the remark following Theorem 1.1). But an irreducible lower separable representation is not necessarilly minimal too (cf. [6]). In the first section, we describe a way to construct a minimal lower separable representation out of an irreducible one. In Section 2 we study the connection between two minimal lower separable representation of a semi-separable kernel and in the third section the number of such representations of a kernel. Both Section 1 and Section 3 are dealing with the questions posed in Section 6 of [6]. The final section concerns the consequences for time varying linear systems with separable boundary conditions.

Finally we make some remarks about kernels of finite rank integral operators

$$K\colon L_2(\Omega, Z) \to L_2(\Sigma, Y),$$

where Ω and Σ are Lebesgue measureable areas in $\mathbb{R}$ with strictly positive Lebesgue measure. If K has finite rank, its kernel k, say, is called a *finite rank* kernel. In that case, k has a separable representation $\{F, G\}$, i.e.,

$$k(t, s) = F(t)\, G(s), \qquad (t, s) \in \Sigma \times \Omega.$$

As before, irreducibility and minimality of representations of k can be defined, but in this case they are equivalent properties. Further, $\{F, G\}$ is minimal if and only if the operators

$$\int_\Sigma F(t)^* F(t)\, dt,$$

$$\int_\Omega G(s)\, G(s)^*\, ds$$

are invertible.

The results in the sequel are all for minimal lower separable representations. By making some adjustments, it's easy to rewrite them in terms of minimal upper separable representations.

1. Construction of a Minimal Lower Separable Representation

1.1. *The Problem*

When a kernel k has a lower separable representation it is possible to reduce it to an irreducible one $\{F, G\}$, say. But, as mentioned in the Introduction, $\{F, G\}$ need not be minimal. Our aim is to construct a minimal lower separable representation, given an irreducible lower separable representation.

1.2. *The Method*

After extending the notion minimal representation, we search for such a representation of k on a part of the lower triangle of $[a, b] \times [a, b]$. This part is enlarged systematically and we find after a finite number of steps a minimal lower separable representation of k.

1.3. *Elementary Parts of the Lower Triangle*

In this section, L is the lower triangle of the square $[a, b] \times [a, b]$, i.e.,

$$L := \{(t, s) \in [a, b] \times [a, b] \mid t > s\}$$

and for γ in (a, b), we define

$$k_\gamma := k|_{[\gamma, b] \times [a, \gamma]}.$$

As mentioned, $\{F, G\}$ is an irreducible representation of k. As before X is the internal space. Its dimension is denoted with n. Define for γ in (a, b) the operators $\Lambda_\gamma: X \to L_2([\gamma, b], Y)$ and $\Gamma_\gamma: L_2([a, \gamma], Z) \to X$ by

$$(\Lambda_\gamma x)(t): F(t)\, x,$$

$$\Gamma_\gamma(\varphi) := \int_a^\gamma G(s)\, \varphi(s)\, ds.$$

We now construct for k elementary parts of L, whereon we can easily classify the separable representations of k. We need the points from the next definition.

DEFINITION 1.1. (a) The point a_i is the unique point in $(a, b]$ with the property: the rank of Γ_γ is greater than or equal to i if $\gamma > a_i$ and smaller than i if $\gamma \leqslant a_i$.

(b) The point b_i is the unique point in $[a, b)$ with the property: the rank of Λ_j is greater than or equal to i if $\gamma < b_i$ and smaller than i if $\gamma \geqslant b_i$.

(c) Put $a_0 = b_{n+1} = a$ and $b_0 = a_{n+1} = b$.

Remark. The points a_i and b_i, $i = 0, 1, ..., n+1$, are well defined. To see this, we consider the functions

$$p, q: (a, b) \to \{0, 1, ..., n\},$$

defined as

$$p(\gamma) := \text{rank } \Lambda_\gamma,$$

$$q(\gamma) := \text{rank } \Gamma_\gamma.$$

The function p is monotonically decreasing and right continuous, while q is monotonically increasing and left continuous. We prove the former assertion. It is clear that Ker $\Lambda_\gamma \subset$ Ker $\Lambda_{\gamma'}$ if $\gamma < \gamma'$. So, p is monotonically decreasing, because

$$p(\gamma) = n - \dim \operatorname{Ker} \Lambda_\gamma.$$

To show that p is right continuous in $\gamma_0 \in (a, b)$, we prove that there exist a $\gamma_1 \in (a, b)$ such that $\gamma_1 > \gamma_0$ and $p(\gamma) = p(\gamma_0)$ for every $\gamma \in (\gamma_0, \gamma_1)$. First, we see that there exist a γ_1, strictly greater than γ_0, such that

$$p(\gamma_1) = \max\{p(\gamma) \mid \gamma > \gamma_0\}.$$

Because $p(\gamma) \leqslant p(\gamma_0)$ for $\gamma > \gamma_0$ and the set Im p is finite. But if

$$F(t)\, x = 0, \qquad \text{for } \gamma < t \leqslant b, \text{ a.e.}$$

for each $\gamma > \gamma_0$, then also

$$F(t)\, x = 0, \qquad \text{for } \gamma_0 < t \leqslant b, \text{ a.e.}$$

and thus

$$\bigcap_{\gamma < \gamma_0} \operatorname{Ker} \Lambda_\gamma \subset \operatorname{Ker} \Lambda_{\gamma_0}.$$

Since

$$\bigcap_{\gamma < \gamma_0} \operatorname{Ker} \Lambda_\gamma \supset \operatorname{Ker} \Lambda_{\gamma_1},$$

it follows that $p(\gamma_1) \geqslant p(\gamma_0)$, hence $p(\gamma_1) = p(\gamma_0)$, which proves the assertion.

A more general version of this remark is given in the proof of Lemma 3.1 of [10].

We order the points a_i and b_i, $i = 0, ..., n+1$, and call them $c_0, c_1, ..., c_{l+1}$, where $c_0 = a$ and $c_{l+1} = b$.

The elementary parts of L are

$$L \cap [c_i, b] \times [a, c_{i+1}], \qquad i = 0, 1, ..., l.$$

1.4. *Representation on the Elementary Parts*

First, we extend the notion minimal separable representation.

DEFINITION 1.2. Let Ω be a Lebesgue measurable area in $\mathbb{R}^2$ with strict positive measure. The pair $\{F, G\}$ is a minimal separable representation of k on Ω if and only if

(i) $k(t, s) = F(t) \cdot G(s)$, for $(t, s) \in \Omega$ a.e., i.e., $\{F, G\}$ is a separable representation of k

(ii) the order of $\{F, G\}$ is minimal in the set of separable representations of k.

We consider the separable representations of k on $\Omega = L \cap [c_i, b] \times [a, c_{i+1}]$ departing from the given irreducible representation $\{F, G\}$ of k on L. Put Ker $\Lambda_{c_i} = X_1$, let X_0 be the orthogonal complement of Ker $\Lambda_{c_i} \cap$ Im $\Gamma_{c_{i+1}}$ in Im $\Gamma_{c_{i+1}}$ and let X_2 be the orthogonal complement of $X_1 \oplus X_0$ in X. With respect to the decomposition $X = X_1 \oplus X_0 \oplus X_2$

$$F_{c_i} = [0 \quad F_0 \quad *],$$

$$G_{c_{i+1}} = \begin{bmatrix} * \\ G_0 \\ 0 \end{bmatrix},$$

where, for $\delta \in (a, b)$, F_δ and G_δ are defined by

$$F_\delta = F|_{[\delta, b]}$$

$$G_\delta = G|_{[a, \delta]}.$$

Then, the pair $\{F_0, G_0\}$ is a separable representation of k on Ω. But for γ in (c_i, c_{i+1}) we have that rank $\Lambda_\gamma =$ rank Λ_{c_i} and rank $\Gamma_\gamma =$ rank $\Gamma_{c_{i+1}}$, hence Ker $\Lambda_\gamma =$ Ker Λ_{c_i} and Im $\Gamma_\gamma =$ Im $\Gamma_{c_{i+1}}$. Analogously as before we define the operators $\Lambda_{0\gamma}: X_0 \to L_2([\gamma, b], Y)$ and $\Gamma_{0\gamma}: L_2([a, \gamma], Z) \to X_0$ as

$$(\Lambda_{0\gamma}(x))(t) = F_0(t)\, x,$$

$$\Gamma_{0\gamma}\, \varphi = \int_a^\gamma G_0(t)\, \varphi(t)\, dt.$$

Because of the construction Ker $\Lambda_{0\gamma} = 0$ and Im $\Gamma_{0\gamma} = X_0$ and it follows that $\{F_{0\gamma}, G_{0\gamma}\}$ is a minimal separable representation of k_γ. The order of $\{F_0, G_0\}$ and the order of $\{F_{0\gamma}, G_{0\gamma}\}$ are equal, so $\{F_0, G_0\}$ is a minimal separable representation of k on Ω. It is also the only one up to similarity. Suppose that $\{F_0', G_0'\}$ is another minimal separable representation of k on Ω, then $\{F_{0\gamma}, G_{0\gamma}\}$ and $\{F_{0\gamma}', G_{0\gamma}'\}$ are similar for every γ in (c_i, c_{i+1}) and we call the similarity operator S. For $\gamma_1 < \gamma_2$, both in (c_i, c_{i+1}) and $t > \gamma_2$ we have

$$\begin{aligned} F_0(t)\, S_{\gamma_1} &= F_0(t)\, S_{\gamma_2} \\ &= F_0'(t). \end{aligned}$$

By multiplying with $F_0(t)^*$, integrating from γ_2 to b, and using the invertibility of

$$\int_\gamma^b F_0(t)^*\, F_0(t)\, dt.$$

we get $S_{\gamma_1} = S_{\gamma_2}$, so S_γ does not depend on γ and the representations $\{F_0, G_0\}$ and $\{F_0', G_0'\}$ are similar.

1.5. *First Step*

The first step is to construct a minimal separable representation of k on $L \cap [a, b] \times [a, c_1]$ with the procedure as described in 1.4.

1.6. *Induction Step*

We use the following notations:

$$A = [a, b] \times [a, c_i] \cap L,$$
$$B = [c_i, b] \times [a, c_{i+1}] \cap L,$$
$$C = A \cap B.$$

The pair $\{F_A, G_A\}$ is a minimal separable representation of k on A. When restricted to C, $\{F_A, G_A\}$ is a dilation of a minimal separable representation of k on C, $\{F_C, G_C\}$, say. Hence

$$F_A = [F_A' \quad F_A''],$$
$$G_A = \begin{bmatrix} G_A' \\ G_A'' \end{bmatrix},$$

where $F_A'(t) = F_C(t)$ for $t > c_i$ a.e., $F_A''(t) = 0$ for $t < c_i$ a.e., and $G_A'(s) = G_C(s)$ a.e.

With the procedure of 1.4 and if necessary with using a similarity we construct a minimal separable representation of k on B, $\{F_B, G_B\}$, such that

$$F_B = [F_B' \quad F_B''],$$
$$G_B = \begin{bmatrix} G_B' \\ G_B'' \end{bmatrix},$$

where $F_B'(t) = F_C(t)$ a.e., $G_B'(s) = G_C(s)$ for $s < c_i$ a.e., and $G_B'(s) = 0$ for $s < c_i$ a.e.

We call the order of $\{F_A, G_A\}$, $\{F_B, G_B\}$, and $\{F_C, G_C\}$, n_A, n_B, and n_C, respectively. We extend F_B'' and G_A'' to $[a, b]$, respectively $[a, c_{i+1}]$, by putting them equal to zero on $[a, c_i)$, respectively $(c_i, c_{i+1}]$. We glue the representations on A and B together:

$$\hat{F} := [F_A' \quad F_A'' \quad F_B''],$$

$$\hat{G} := \begin{bmatrix} G_B' \\ G_A'' \\ G_B'' \end{bmatrix}.$$

We obtain a separable representation $\{\hat{F}, \hat{G}\}$ of k on $A \cup B$, with order $n_A + n_B - n_C$. It is also minimal. If not, there would exist another representation $\{\tilde{F}, \tilde{G}\}$ with order $q < n_A + n_B - n_C$.

If necessary, we take a representation similar to $\{\tilde{F}, \tilde{G}\}$, such that, when restricted to B, it's a dilation of $\{F_B, G_B\}$. Hence we may suppose that

$$\tilde{F} = [\tilde{F}' \quad \tilde{F}''],$$

$$\tilde{G} = \begin{bmatrix} G_B \\ \tilde{G}'' \end{bmatrix},$$

where $\tilde{F}'(t) = F_B(t)$ for $t > c_i$ a.e. But

$$G_B(s) = \begin{bmatrix} G_C(s) \\ 0 \end{bmatrix}, \qquad \text{for} \quad s < c_i \text{ a.e.},$$

so, when $\{\tilde{F}, \tilde{G}\}$ is restricted to A and the zero part of G_B plus the corresponding part of $\tilde{F}$ is deleted, we become a representation of k on A with order $q + n_c - n_B$. This number is smaller than n_A, which is impossible. We may conclude that $\{\hat{F}, \hat{G}\}$ is a minimal separable representation of k on $A \cup B$.

Starting with the minimal separable representation on $[a, b] \times [a, c_1] \cap L$, we can apply the procedure, as described above, successively for $i = 1, 2, ..., l$. In the last step, $i = l$, we obtain $A \cup B = L$. So, the corresponding $\{\hat{F}, \hat{G}\}$ is a minimal lower separable representation of k.

Remark. Now, we are able to compute the number $l(k)$, the lower order of k. Therefore, we adjust the used notations as

$$A_i = [a, b] \times [a, c_i] \cap L,$$

$$B_i = [c_i, b] \times [a, c_{i+1}] \cap L,$$

$$C_i = A_i \cap B_i, \qquad i = 1, ..., l.$$

As above the numbers n_{A_i}, n_{B_i}, and n_{C_i} are the orders of minimal separable representations on the areas A_i, B_i, and C_i, respectively. It is easy to see that

$$l(k) = \left(n_{A_1} + \sum_{i=1}^{l} (n_{B_i} - n_{C_i}) \right).$$

This expression is a special case of a general "lower order" formula, which is derived by H. J. Woerdeman (cf. [12]).

1.7. *Example*

To illustrate the method, introduced in this section we give an example. Put $L = \{(t, s) \in [0, 1] \times [0, 1] \mid t > s\}$ and

$$F(t)=\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}, \qquad 0\leqslant t<\frac{1}{2},$$

$$=\begin{bmatrix}1&1&1\\0&1&1\\0&0&0\end{bmatrix}, \qquad \frac{1}{2}\leqslant t\leqslant 1,$$

$$G(s)=\begin{bmatrix}1&1&1\\0&1&1\\0&1&1\end{bmatrix}, \qquad 0\leqslant s\leqslant\frac{1}{2},$$

$$=\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}, \qquad \frac{1}{2}<s\leqslant 1.$$

We consider the kernel

$$k(t,s)=F(t)\,G(t), \qquad 0\leqslant s<t\leqslant 1,$$
$$=0, \qquad 0\leqslant t<s\leqslant 1.$$

The pair $\{F, G\}$ is an irreducible lower separable representation of k. The number l equals 1 and the points c_i, $i=0, 1, 2$, are 0, $\frac{1}{2}$, and 1, while the areas A, B, and C are given by

$$A=[0,1]\times\left[0,\frac{1}{2}\right]\cap L,$$

$$B=\left[\frac{1}{2},1\right]\times[0,1]\cap L,$$

$$C=\left[\frac{1}{2},1\right]\times\left[0,\frac{1}{2}\right].$$

The corresponding minimal separable representations can be computed:

$$F_A(t)=\begin{bmatrix}1&2\\0&2\\0&1\end{bmatrix}, \qquad 0\leqslant t<\frac{1}{2},$$

$$=\begin{bmatrix}1&2\\0&2\\0&0\end{bmatrix}, \qquad \frac{1}{2}\leqslant t\leqslant 1,$$

$$G_A(s)=\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 1\end{bmatrix}, \qquad 0\leqslant s\leqslant\frac{1}{2},$$

$$F_B(t)=F_C(t)=\begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{bmatrix}, \qquad \frac{1}{2}\leqslant t\leqslant 1,$$

$$G_B(s)=\begin{bmatrix}1 & 1 & 1\\ 0 & 2 & 2\end{bmatrix}, \qquad 0\leqslant s\leqslant\frac{1}{2},$$

$$=\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 2\end{bmatrix}, \qquad \frac{1}{2}< s\leqslant 1,$$

$$G_C(s)=\begin{bmatrix}1 & 1 & 1\\ 0 & 2 & 2\end{bmatrix}, \qquad 0\leqslant s\leqslant\frac{1}{2}.$$

The pair $\{F_A S, S^{-1}G_A\}$, where S is given by

$$S=\begin{bmatrix}1 & 0\\ 0 & \frac{1}{2}\end{bmatrix},$$

restricted to C, is a dilation of $\{F_C, G_C\}$, because $F_A S|_{[1/2,1]}=F_C$ and $S^{-1}G_A=G_C$.

Finally, we are able to construct a minimal lower separable representation $\{F, G\}$ with the procedure of this section:

$$\hat{F}(t)=\begin{bmatrix}1 & 1\\ 0 & 1\\ 0 & \frac{1}{2}\end{bmatrix}, \qquad 0\leqslant t<\frac{1}{2},$$

$$=\begin{bmatrix}1 & 1\\ 0 & 1\\ 0 & 0\end{bmatrix}, \qquad \frac{1}{2}\leqslant t\leqslant 1,$$

$$\hat{G}(s)=\begin{bmatrix}1 & 1 & 1\\ 0 & 2 & 2\end{bmatrix}, \qquad 0\leqslant s\leqslant\frac{1}{2},$$

$$=\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 2\end{bmatrix}, \qquad \frac{1}{2}< s\leqslant 1.$$

The lower order of k is two.

2. Classification of Minimal Lower Separable Representations of a Kernel

In this section, we study the set of minimal lower separable representations of a kernel k. If all the minimal lower separable representations of k

are similar, k is called *lower unique* (e.g., kernels with uniform lower order, cf. [6, Sect. 5]). In the general case we will also give a link between two such representations with the help of an invertible operator, only, the description is more complicated.

Let $\{F, G\}$ be a minimal lower separable representation of k, let X be the internal space, and let n be the lower order of k. We define two sets of subspaces of X, which will be used in the discussion. The operators Λ_γ, Γ_γ, $\gamma \in (a, b)$, and the points a_i, b_i, $i = 0, ..., n+1$, are as in the first section. For our convenience, we use the notations

$$\Lambda_j := \Lambda_{b_{j+1}}, \qquad j = 0, ..., n,$$
$$\Gamma_i := \Gamma_{a_{i+1}}, \qquad i = 0, ..., n.$$

The spaces X_i, $i = 0, ..., n$, are defined as

$$X_0 := \{0\},$$
$$X_i := X_{i-1} \quad \text{if} \quad a_i = a_{i+1},$$
$$X_i := \operatorname{Im} \Gamma_i \ominus \operatorname{Ker} \Lambda_j \quad \text{if} \quad a_i \neq a_{i+1},$$

where in the last equation j is the smallest number such that $b_{j+1} \leqslant a_i$. The spaces Y_j, $j = 0, ..., n$, are defined as

$$Y_0 := \{0\},$$
$$Y_j := Y_{j-1} \quad \text{if} \quad b_j = b_{j+1},$$
$$Y_j := \operatorname{Im} \Gamma_i \ominus \operatorname{Ker} \Lambda_j \quad \text{if} \quad b_j \neq b_{j+1},$$

where in the last equation i is the smallest number such that $a_{i+1} \geqslant b_j$. We denote with P_i, respectively Q_i, the orthogonal projection of X on X_i, respectively Y_i, $i = 0, ..., n$.

To verify that these spaces are well-defined, the next lemma is necessary.

LEMMA 2.1. *Let* $0 \leqslant i \leqslant n$ *and* $0 \leqslant j \leqslant n$.
If $b_{j+1} \leqslant a_{i+1}$, *then* $\operatorname{Ker} \Lambda_j \subset \operatorname{Im} \Gamma_i$.

Proof. Let

$$X_1 = \operatorname{Im} \Gamma_i$$
$$X_2 = \operatorname{Ker} \Lambda_j \ominus \operatorname{Im} \Gamma_i \cap \operatorname{Ker} \Lambda_j$$

and let X_3 be a subspace of X, such that

$$X = X_1 \oplus X_2 \oplus X_3.$$

We rewrite F and G, relative to this decomposition:

$$F = [F_1 \quad F_2 \quad F_3],$$

$$G = \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix}.$$

Because of the construction, we have that

$$F_2(t) = 0, \qquad \text{for } b_{j+1} \leqslant t \leqslant b, \text{ a.e.},$$

$$G_2(s) = 0, \qquad \text{for } a \leqslant s \leqslant a_{i+1}, \text{ a.e.},$$

and hence

$$F_2(t) \cdot G_2(s) = 0 \qquad \text{for } a \leqslant s < t \leqslant b, \text{ a.e.}$$

If X_2 is not the zero space, then the pair

$$\left\{ [F_1 \quad F_3], \begin{bmatrix} G_1 \\ G_3 \end{bmatrix} \right\}$$

is a lower separable representation of k, with order strictly smaller than the order of $\{F, G\}$. But this is impossible, because $\{F, G\}$ has minimal order, hence

$$\operatorname{Ker} \Lambda_j \subset \operatorname{Im} \Gamma_i. \quad \blacksquare$$

If $\{F_0, G_0\}$ is another minimal lower separable representation of k and X_0 its internal space, one can define the maps Λ_{0i}, Γ_{0i}, the spaces X_{0i}, Y_{0i}, and the projections P_{0i}, Q_{0i}, $i = 0, ..., n$, analogously as for $\{F, G\}$. In the sequel, we introduce other spaces and maps connected with $\{F, G\}$. The analogous notations for the pair $\{F_0, G_0\}$ will not be defined, but when we use them, we add a nought to the original notation.

The maps Λ_γ, Γ_γ, $\Lambda_{0\gamma}$, $\Gamma_{0\gamma}$, $\gamma \in (a, b)$, have the surprising property that

$$\operatorname{rank} \Lambda_\gamma = \operatorname{rank} \Lambda_{0\gamma},$$

$$\operatorname{rank} \Gamma_\gamma = \operatorname{rank} \Lambda_{0\gamma},$$

for each γ in (a, b). This is proven by H. J. Woerdeman for a more general case in [12, Sect. 6]. It follows that the points a_i, b_i, $i = 0, ..., n+1$, are independent of the chosen minimal lower separable representation.

We are now able to state the main result of this section.

THEOREM 2.2. *Let $\{F, G\}$ and $\{F_0, G_0\}$ be minimal lower separable representations of a kernel k, with internal spaces X, respectively X_0. Let the* *spaces* X_i, X_{0i}, Y_i, Y_{0i} *and the projections* P_i, P_{0i}, Q_i, Q_{0i}, $i=0, ..., n$, *be defined as above.*

There exists an invertible operator $R: X_0 \to X$, *such that*

$$F_0(t)|_{X_{0i}} = F(t)|_{X_i}(P_i R|_{X_{0i}}), \qquad a_i \leqslant t \leqslant b, \text{ a.e., } i=0, ..., n; \tag{2.1}$$

$$Q_{0i} G_0(s) = (Q_i R|_{Y_{0i}})^{-1} Q_i G(s), \qquad a \leqslant s \leqslant b_i, \text{ a.e., } i=0, ..., n. \tag{2.2}$$

Before proving the theorem we give some partial results in four lemmas.

LEMMA 2.3. *Let* $\{F, G\}$ *and* $\{F_0, G_0\}$ *be minimal lower separable representations of a kernel k and let* X_i, X_{0i}, P_i, P_{0i}, $i=1, ..., n$, *be defined as above. There exists an invertible operator* $R_i: X_{0i} \to X_i$ *such that*

$$F_0(t)|_{X_{0i}} = F(t)|_{X_i} R_i, \qquad a_i \leqslant t \leqslant b, \text{ a.e.,} \tag{2.3}$$

$$P_{0i} G_0(s) = R_i^{-1} P_i G(s), \qquad a \leqslant s \leqslant \min\{a_{i+1}, b_j\}, \text{ a.e.,} \tag{2.4}$$

where j is the smallest number such that $b_{j+1} \leqslant a_i$.

Proof. From the method as described in part 1.4 of this paper, it follows that

$$F_0(t)|_{X_{0i}} = F(t)|_{X_i} R_{i\delta}, \qquad \text{for } \delta \leqslant t \leqslant b, \text{ a.e.,}$$

$$P_{0i} G_0(s) = R_{i\delta}^{-1} P_i G(s), \qquad \text{for } a \leqslant s \leqslant \delta, \text{ a.e.,}$$

where $\delta \in (a_i, \min\{a_{i+1}, b_j\})$ and $R_{i\delta}$ is an invertible operator. Again as in part 1.4 one shows that $R_{i\delta}$ does not depend on δ, which proves the lemma.

The next lemma is similar to Lemma 2.4. So, we can delete the proof.

LEMMA 2.4. *Let* $\{F, G\}$ *and* $\{F_0, G_0\}$ *be minimal lower separable representations of a kernel k and let* Y_j, Y_{0j}, Q_j, Q_{0j}, $j=1, ..., n$, *be defined as above. There exists an invertible operator* $S_j: Y_j \to Y_{0j}$, *such that*

$$Q_{0j} G_0(s) = S_j Q_j G(s), \qquad a \leqslant s \leqslant b_j, \textit{ a.e.,} \tag{2.5}$$

$$F_0(t)|_{Y_{0j}} = F(t)|_{Y_j} S_j^{-1}, \qquad \max\{b_{j+1}, a_i\} \leqslant t \leqslant b, \textit{ a.e.,} \tag{2.6}$$

where i is the smallest number such that $a_{i+1} \geqslant b_j$.

With the notations of Lemma 2.3 and Lemma 2.4, Theorem 2.2 says that there exist an invertible operator R such that

$$P_i R|_{X_{0i}} = R_i,$$

$$(Q_i R|_{Y_{0i}})^{-1} = S_i$$

for $i=1, \ldots, n$. In our proof we construct the operator R by sticking the operators R_i, $i=1, \ldots, n$, together. For this purpose we use the fact that the restriction of two operators R_i, R_j, $1 \leqslant i, j \leqslant n$, to the common part of their domains, are equal up to a projection in some cases. This is the subject of Lemma 2.5.

LEMMA 2.5. *Let $1 \leqslant i \leqslant j \leqslant n$, such that there exists a number m, with $b_{m+1} \leqslant a_j < b_m$ and $b_{m+1} \leqslant a_{i+1}$ and let R_i, R_j as in Lemma 2.3. Let $D = X_i \cap X_j$, $D_0 = X_{0i} \cap X_{0j}$ and let P be the orthogonal projection of X_i on D. Then*

$$PR_i|_{D_0} = R_j|_{D_0}. \tag{2.7}$$

Proof. From Lemma 2.3, we know that

$$F_0(t)|_{X_{0i}} = F(t)|_{X_i} R_i, \qquad a_i \leqslant t \leqslant b, \text{ a.e.,}$$
$$F_0(t)|_{X_{0j}} = F(t)|_{X_j} R_j, \qquad a_j \leqslant t \leqslant b, \text{ a.e.}$$

By restricting both members of both equations to D_0, we get that

$$F(t)|_{X_i} R_i|_{D_0} = F(t)|_{X_j} R_j|_{D_0}, \qquad a_j \leqslant t \leqslant b, \text{ a.e.} \tag{2.8}$$

Since $b_{m+1} \leqslant a_j < b_m$, we get

$$X_j = \operatorname{Im} \Gamma_j \ominus \operatorname{Ker} \Lambda_m.$$

We have also that $b_{m+1} \leqslant a_{i+1}$, so, $\operatorname{Ker} \Lambda_m \subset \operatorname{Im} \Gamma_i$ and hence

$$D = \operatorname{Im} \Gamma_i \ominus \operatorname{Ker} \Lambda_m.$$

Since $X_i \subset \operatorname{Im} \Gamma_i$, we obtain

$$X_i \ominus D \subset \operatorname{Ker} \Lambda_m,$$

thus $F(t)|_{X_i}(I-P)=0$ for $a_j \leqslant t \leqslant b$, a.e. It follows that

$$\begin{aligned} F(t)|_{X_i} R_i|_{D_0} &= F(t)|_{X_i} PR_i|_{D_0} \\ &= F(t)|_D PR_i|_{D_0} \\ &= F(t)|_{X_j} PR_i|_{D_0}, \qquad a_j \leqslant t \leqslant b, \text{ a.e.} \end{aligned}$$

With this result and (2.8), we get that

$$F(t)|_{X_j} PR_i|_{D_0} = F(t)|_{X_j} R_j|_{D_0}, \qquad a_j \leqslant t \leqslant b, \text{ a.e.}$$

Using the same method as at the end of part 1.4, we obtain (2.10) from this equation and from the injectivity of the operator

$$\Lambda_{a_j}|_{X_j}. \quad \blacksquare$$

The next lemma can be proved analogously.

LEMMA 2.6. *Let* $1 \leqslant i, j \leqslant n$ *be such that* $a_i < b_j \leqslant a_{i+1}$, *Let* R_i, *respectively* S_j, *be as in Lemma* 2.3, *respectively* 2.4, *then*

$$S_j^{-1} = Q_j R_i|_{Y_{0j}}.$$

Proof of Theorem 2.2. The operator is constructed inductively. For $j = 1, ..., n$, we denote with $i(j)$ the largest number such that $a_{i(j)} < b_j$. We put

$$B_n := \operatorname{Im} \Gamma_{i(n)}$$

and we define the operator $T_n \colon B_{0n} \to B_n$ as

$$T_n := R_{i(n)}.$$

Clearly, T_n is invertible. For $j = n-1, ..., 1$, we consider two cases. If $b_{j+1} > a_{i(j)}$, then we put

$$A_j := \{0\},$$
$$B_j := B_{j+1},$$
$$T_j := T_{j+1}.$$

If $b_{j+1} \leqslant a_{i(j)}$, we define

$$A_j := X_{i(j)} \ominus B_{j+1} \cap X_{i(j)},$$
$$B_j := B_{j+1} \oplus A_j.$$

The operator P_0 denotes the projection of B_{0j} on B_{0j+1} along A_{0j} and the operator $T_j \colon B_{0j} \to B_j$ is defined as

$$T_j x_0 := T_{j+1} P_0 x_0 + R_{i(j)}(I - P_0)\, x_0,$$

where $x_0 \in B_{0j}$ and finally

$$R := T_1.$$

First we want to prove that the domain of R, which is B_{01}, equals the whole space X_0. We will show that

$$B_{0j} = \operatorname{Im} \Gamma_{0i(j)} \tag{2.9}$$

for $j=n, \ldots, 1$ and hence

$$B_{01}=\operatorname{Im}\Gamma_{0i(1)}=\operatorname{Im}\Gamma_{0n}=X_0.$$

The second equation follows from the fact that $a_n<b_1$, because if not, the lower order of k would be strictly smaller than n.

Equation (2.9) is true for $j \times n$, because of the definition B_n. Suppose that (2.9) is true for an arbitrary j, $1<j\leqslant n$, then

$$B_{0j-1}=B_{0j}$$

if $b_j>a_{i(j-1)}$ and if $b_j\leqslant a_{i(j-1)}$ we have

$$B_{0j-1}=B_{0j}+X_{0i(j-1)}.$$

In the former case $a_{i(j-1)}=a_{i(j)}$. So, (2.9) is true for $j-1$. In the latter case, since

$$X_{0i(j-1)}=\operatorname{Im}\Gamma_{0i(j-1)}\ominus\operatorname{Ker}\Lambda_{0j-1}$$

and we know from Lemma 2.1 that

$$\operatorname{Ker}\Lambda_{0j-1}\subset\operatorname{Im}\Gamma_{0i(j)}=B_{0j},$$

because $b_j\leqslant a_{i(j)+1}$, we obtain that (2.9) is true for $j-1$. This proves Eq. (2.9) for $j=1, \ldots, n$.

Next we will prove that R is invertible, by showing that T_j is injective for each $j=1, \ldots, n$. Surely, T_n is invertible, thus injective. Suppose that T_j is injective for a number j, $1<j\leqslant n$. If $b_j>a_{i(j-1)}$, then it follows immediately that T_{j-1} is injective. So, let $b_j\leqslant a_{i(j-1)}$ and let $x_0\in B_{0j-1}$, such that $T_{j-1}x_0=0$, i.e.,

$$T_jP_0x_0+R_{i(j-1)}(I-P_0)\,x_0=0,$$

where P_0 is the projection of B_{0j-1} on B_j along A_{0j-1}. If we can prove that $(I-P_0)\,x_0=0$, then $T_jP_0x_0=0$ and from the hypothesis, it would follow that $P_0x_0=0$, hence $x_0=0$. Let P be the projection of B_{j-1} on B_j along A_{j-1} and let $C=X_{i(j)}\cap X_{i(j-1)}$, then

$$0=(I-P)\,T_{j-1}x_0=(I-P)\,R_{i(j-1)}(I-P_0)\,x_0$$

and from Lemma 2.5 it follows that

$$\operatorname{Im} R_{i(j-1)}|_{C_0}=C.$$

So, there exists an element of C_0, x_1, say, such that

$$R_{i(j-1)}x_1=-P_CR_{i(j-1)}(I-P_0)\,x_0,$$

where P_C is the orthogonal projection of $X_{i(j-1)}$ on C. But then we have that

$$R_{i(j-1)}(x_1 + (I - P_0)\, x_0) = 0$$

and hence $x_1 = (I - P_0)\, x_0 = 0$.

We will show Eq. (2.1), by first proving the following subresult:

$$\tilde{P}_j T_j |_{X_{0i(j)}} = R_{i(j)}, \qquad j = 1, ..., n. \tag{2.10}$$

Here $\tilde{P}_j$ is the orthogonal projection of B_j on $X_{i(j)}$. Clearly (2.10) is true for $j = n$. Suppose that (2.10) is true for a number l, $1 < l \leqslant n$. If $b_l > a_{i(l-1)}$, then (2.10) for $j = l-1$, follows trivially. So, we assume that $b_l \leqslant a_{i(l-1)}$. Let $x_0 \in X_{0i(l-1)}$ and let P_0 be the projection of B_{0l-1} on B_{0l} along A_{l-1}. From the definition of T_{l-1} we get that

$$\tilde{P}_{l-1} T_{l-1} x_0 = \tilde{P}_{l-1} T_l P_0 x_0 + R_{i(l-1)}(I - P_0)\, x_0. \tag{2.11}$$

We also have that $P_0 x_0 \in X_{0i(l-1)}$, because $(I - P_0)\, x_0 \in X_{0i(l-1)}$. For $x \in B_l$, we may write $x = x_1 + x_2$ relative to the decomposition $B_l = (B_l \ominus \operatorname{Ker} A_{l-1}) \oplus \operatorname{Ker} A_{l-1}$. But $\operatorname{Ker} A_{l-1}$ is orthogonal to $X_{i(l-1)}$, hence

$$\tilde{P}_{l-1} x = x_1 = \tilde{Q} x,$$

where $\tilde{Q}$ is the orthogonal projection of B_l on the space

$$X_{i(l-1)} \cap X_{i(l)} = X_{i(l-1)} \cap B_l = \operatorname{Im} \Gamma_{i(l)} \ominus \operatorname{Ker} A_{l-1}.$$

We factorize $\tilde{Q}$ as follows: $\tilde{Q} = \tilde{Q}_1 \cdot \tilde{Q}_2$, where $\tilde{Q}_2$ is the orthogonal projection of B_l on $X_{i(l)}$ and $\tilde{Q}_1$ is the orthogonal projection of $X_{i(l)}$ on $X_{i(l)} \cap X_{i(l-1)}$. Then

$$\tilde{Q}_1 \cdot \tilde{Q}_2 T_l P_0 x_0 = \tilde{Q}_1 R_{i(l)} P_0 x_0 = R_{i(l-1)} P_0 x_0,$$

because of the induction-hypothesis and Lemma 2.5. If we use this result in (2.11) we get (2.10) for $j = l-1$. Now, suppose that $1 \leqslant i \leqslant n$ and $a_{i(j+1)} < a_i \leqslant a_{i(j)}$. Then $X_i = \operatorname{Im} \Gamma_i \ominus \operatorname{Ker} A_j$ and hence X_i is a subspace of $X_{i(j)}$. We also have that $b_{j+1} \leqslant a_i \leqslant a_{i(j)} < b_j$. So, if we let $\bar{P}$ be the orthogonal projection of $X_{i(j)}$ on X_i, it follows from (2.10) and Lemma 2.5 that

$$\begin{aligned} P_i R |_{X_{0i}} &= P_i T_j |_{X_{0i}} \\ &= \bar{P} \tilde{P}_{i(j)} T_j |_{X_{0i}} \\ &= \bar{P} R_{i(j)} |_{X_{0i}} \\ &= R_i. \end{aligned}$$

Finally, we prove (2.2) briefly. If $1 \leqslant j \leqslant n$ and $b_{j+1} \leqslant a_{i(j)}$, we have that

$$Y_j = X_{i(j)} = \operatorname{Im} \Gamma_{i(j)} \ominus \operatorname{Ker} \Lambda_j$$

and it is easy to see that

$$S_j^{-1} = R_{i(j)} = Q_j R \mid_{Y_{0j}}.$$

If $b_{j+1} > a_{i(j)}$, then we let l be the number such that $a_{i(l)} = a_{i(j)}$, and $b_{l+1} \leqslant a_{i(l)}$. From Lemma 2.6., it follows that

$$S_j^{-1} = Q_j R_{i(l)} \mid_{Y_{0j}}.$$

If $\tilde{P}_{i(l)}$ is as above and $\bar{Q}$ is the orthogonal projection of $X_{i(l)}$ on Y_j, then

$$\begin{aligned} Q_j R \mid_{Y_{0j}} &= Q_j T_l \mid_{Y_{0j}} \\ &= \bar{Q} \tilde{P}_{i(l)} T_l \mid_{Y_{0j}} \\ &= \bar{Q} R_{i(l)} \mid_{Y_{0j}} \\ &= S_j^{-1}, \end{aligned}$$

which proves (2.2) in this case. ∎

From the discussion above, we can derive necessary and sufficient conditions for a pair $\{F_0, G_0\}$ to be a minimal separable representation of a kernel k. We notice that it is always possible to find such a representation for k, with the construction method of Section 1.

COROLLARY 2.7. *Let $\{F, G\}$ be a minimal lower separable representation of a kernel k and let $\{F_0, G_0\}$ be another pair of operator valued functions with internal space X_0 and order n. Let the points $a_i, b_i, i = 0, ..., n+1$, respectively a_{0i}, b_{0i}, $i = 0, ..., n+1$, be as in Definition 1.1 for $\{F, G\}$, respectively $\{F_0, G_0\}$.*

The pair $\{F_0, G_0\}$ is a minimal lower separable representation of k if and only if

(a) *$a_i = a_{0i}$, $b_i = b_{0i}$ for $i = 0, ..., n+1$*

(b) *there exists an invertible operator R: $X_0 \to X$ such that conditions (2.1) and (2.2) are fulfilled.*

Proof. The only if part follows from Theorem 2.2 and from the remark before it.

To prove the if part, we have to show that

$$F_0(t)\, G_0(s) = F(t)\, G(s), \qquad \text{for } a \leqslant s < t \leqslant b, \text{ a.e.}$$

Suppose that $a \leqslant s < t \leqslant b$. Take $\delta \in (s, t)$ and let i and j be the numbers such that $a_i < \delta \leqslant a_{i+1}$ and $b_{j+1} \leqslant \delta < b_j$. With the method as explained in part 1.4, we get

$$F_0(t)\, G_0(s) = F_0(t) |_{\bar{X}_0}\, Q_0 G_0(s).$$

Here $\bar{X}_0 = \operatorname{Im} \Gamma_i \ominus \operatorname{Ker} \Lambda_j$ and Q_0 is the orthogonal projection of X_0 on $\bar{X}_0$. Since $\bar{X}_0 \subset Y_{0j} \subset X_{0i(j)}$, we obtain

$$F_0(t)\, G_0(s) = F_0(t) |_{X_{0i(j)}}\, Q_{0j} G_0(s) = F(t)\, P_{i(j)}\, R |_{X_{0i(j)}} (Q_j R |_{Y_{0j}})^{-1}\, Q_j G(s),$$

where we used (2.1) and (2.2). From Lemma 2.6 it follows that

$$P_{i(j)} R_{X_{0i(j)}} |_{Y_{0j}} = Q_j R |_{Y_{0j}},$$

and hence

$$F_0(t)\, G_0(s) = F(t) |_{X_{i(j)}}\, Q_j G(s) = F(t)\, G(t),$$

where the last equation again follows from the method of part 1.4. ∎

Using the notations of the preceding corollary one proves analogously that a pair $\{F_0, G_0\}$ is a minimal lower separable representation of k if and only if

(a) $a_{0i} = a_i$ and $b_{0i} = b_i$ for $i = 0, ..., n+1$;

(b′) for $i = 1, ..., n$, there exist operators R_i, S_i such that

$$\begin{aligned}
F_0(t) |_{X_{0i}} &= F(t) |_{X_i} R_i, && a_i \leqslant t \leqslant b, \text{ a.e.,}\\
P_{0i} G_0(s) &= R_i^{-1} P_i G(s), && a \leqslant s \leqslant a_i, \text{ a.e.,}\\
Q_{0i} G_0(s) &= S_i Q_i G(s), && a \leqslant s \leqslant b_i, \text{ a.e.,}\\
F_0(t) |_{Y_{0i}} &= F(t) |_{Y_i} S_i^{-1} && b_i \leqslant t \leqslant b, \text{ a.e.}
\end{aligned}$$

This gives us a criterion for $\{F_0, G_0\}$ to be a minimal lower separable representation. First, one has to search for the points wherein the functions $\gamma \to \operatorname{rank}(\Lambda_{0\gamma})$ and $\gamma \to \operatorname{rank}(\Gamma_{0\gamma})$ are discontinuous and compair them with a_i, b_i, $i = 0, ..., n+1$. Then, one has to check $2n$ times whether or not two pairs of operator-valued functions are similar.

3. The Number of Non-similar Minimal Lower Separable Representations

We already know from [6] that a minimal lower separable representation $\{F, G\}$ of k is the only one up to similarity if

$$\int_\gamma^b F(t)^* F(t)\,dt, \tag{3.1}$$

$$\int_a^\gamma G(s)\,G(s)^*\,ds \tag{3.2}$$

are invertible for each γ in (a, b). Then, the kernel k is called lower-unique. We intend to prove the reverse of this statement; i.e., if either one of the operators (3.1) or (3.2) is not invertible for a γ in (a, b), then k is not lower unique. More, k has even uncountable non-similar minimal lower separable representations. We split the proof into two propositions. We use the notations of the Introduction.

PROPOSITION 3.1. *Let $\{F, G\}$ be a minimal lower separable representation of a kernel k. Suppose that for some γ the operator*

$$\int_\gamma^b F(t)^* F(t)\,dt \tag{3.1}$$

is not invertible. Then k has uncountable many non-similar minimal lower separable representations.

Proof. With the notations of the first section, (3.1) equals $\Lambda_\gamma^* \Lambda_\gamma$, so (3.1) is invertible if and only if Λ_γ is injective. Suppose that the operator (3.1) is not invertible for $\gamma = \gamma_0$. Choose a nonzero vector z in Z and let M be a subspace of Z such that $\{z\} \oplus M = Z$. Define for $x \in \operatorname{Ker} \Lambda_{\gamma_0}$ the operator $\phi_x: Z \to X$ by $\phi_x(z) = x$ and $\phi_x(m) = 0$ for $m \in M$. Here X is the internal space of the pair $\{F, G\}$. Further, let $H_x(t): Z \to X$, $a \leqslant t \leqslant b$, be defined by

$$H_x(t) = \begin{cases} 0 & a \leqslant t < \gamma_0, \\ \phi_x & \gamma_0 \leqslant t \leqslant b. \end{cases}$$

Then, since $F(t)\,H_x(s) = 0$ for $a \leqslant s < t \leqslant b$, a.e. and the order of $\{F, G + H_x\}$ equals the order of $\{F, G\}$, we get that $\{F, G + H_x\}$ is a minimal lower separable representation of k. Furthermore, it is easy to check that $\{F, G + H_{x_1}\}$ and $\{F, G + H_{x_2}\}$ are similar if and only if $x_1 = x_2$. So, since $\ker \Lambda_{\gamma_0} \neq \{0\}$ the set $\{\{F, G + H_x\} \mid x \in \ker \Lambda_{\gamma_0}\}$ gives an uncountable set of non-similar minimal lower separable representations of k. ∎

Analogously one proves the following proposition.

PROPOSITION 3.2. *Let $\{F, G\}$ be a minimal lower separable representation of a kernel k. Suppose that for some γ the operator*

$$\int_a^\gamma G(s)\,G(s)^*\,ds \tag{3.2}$$

is not invertible. Then k has uncountable many non-similar minimal lower separable representations.

Another characterization of lower uniqueness can be found in [10], Theorem 0.1, where it was showed that the kernel k is lower unique if and only if for each γ in (a, b) the rank of k_γ is independent of γ (cf. part 1.3 for the definition of k_γ).

4. SB Systems

A motivation for researching semi-separable kernels is the study of minimalization of linear time-varying systems. In this section, we give the corresponding consequences for *linear time-varying systems with separable boundary conditions* (SB systems) (cf. [6, Sect. 7]). In the state space representation an SB system has the following form:

$$\theta \begin{cases} \dot{x}(t) = A(t)\, x(t) + B(t)\, u(t), & a \leqslant t \leqslant b, \\ y(t) = C(t)\, x(t) + D(t)\, u(t), & a \leqslant t \leqslant b, \\ (I - P)\, x(a) = 0, & PU(b)^{-1}\, x(b) = 0, \end{cases}$$

which we shall abbreviate by

$$\theta = (A(t), B(t), C(t), D(t), P)_a^b.$$

Here $u \in L_2([a, b], Z), y \in L_2([a, b], Y)$, x is an absolute continuous function, with values in X, where the spaces X, Y, Z and $L_2([a, b], U)$ are as above. Further, for each t in $[a, b]$

$$A(t)\colon X \to X, \qquad B(t)\colon Z \to X, \qquad C(t)\colon X \to Y, \qquad D(t)\colon Z \to Y$$

are operators where the entries of (the matrix of) $A(t)$ are integrable, the entries of $B(t)$ and $C(t)$ are square integrable and the entries of $D(t)$ are measurable and essentially bounded on $[a, b]$. The operator-valued function $U(t)$ denotes the *fundamental operator* of the system, i.e., the unique solution of

$$\dot{U}(t) = A(t)\, U(t), \qquad a \leqslant t \leqslant b,$$
$$U(a) = I_X,$$

where I_X is the identity operator on X (cf. [1, 2]). The space X is called the state space of θ.

The operator P is assumed to be a projection on X. This property is expressed in the word "separable" in the description of the boundary conditions. The system θ has a well-defined input–output operator,

$$T_\theta\colon L_2([a, b], Z) \to L_2([a, b]\ Y),$$

which is the integral operator,

$$(T_\theta \varphi)(t) = D(t)\,\varphi(t) + \int_a^b k(t,s)\,\varphi(s)\,ds, \qquad a \leqslant t \leqslant b,$$

where the kernel k is given by

$$k(t,s) = \begin{cases} C(t)\,U(t)(I-P)\,U(s)^{-1}\,B(s), & a \leqslant s < t \leqslant b \\ -C(t)\,U(t)\,PU(s)^{-1}\,B(s), & a \leqslant t < s \leqslant b. \end{cases}$$

Clearly k is semi-separable, which is equivalent to saying that k has a lower and an upper separable representation (see [4, Sect. I.4]). Conversely, any integral operator with a semi-separable kernel is the input–output operator of an SB system (see [4, Sect. I.4]).

If $T = T_\theta$, then θ is called a *realization* of T. An SB system θ is *SB minimal* if among all SB systems with input–output operator T_θ the dimension of the state space of θ is as small as possible.

Two SB systems

$$\theta_1 = (A_1(t),\ B_1(t),\ C_1(t),\ D_1(t),\ P_1)_a^b$$

and

$$\theta_2 = (A_2(t),\ B_2(t),\ C_2(t),\ D_2(t),\ P_2)_a^b.$$

with state spaces X_1 and X_2, are said to be similar if and only if there exist an operator-valued function $S(t)$, such that

$$S(t)\colon X_1 \to X_2$$

is invertible for each t in $[a, b]$, the entries of $S(t)$ are absolute continuous, and

$$\begin{aligned} A_1(t) &= S(t)^{-1}\,A_2(t)\,S(t) + S(t)^{-1}\dot{S}(t), && a \leqslant t \leqslant b, \text{ a.e.},\\ B_1(t) &= S(t)\,B_2(t), && a \leqslant t \leqslant b, \text{ a.e.},\\ C_1(t) &= C_2(t)\,S(t)^{-1}, && a \leqslant t \leqslant b, \text{ a.e.},\\ D_1(t) &= D(t), && a \leqslant t \leqslant b, \text{ a.e.},\\ P_1 &= S(a)^{-1}\,P_2 S(a). \end{aligned}$$

For more information about linear systems and linear systems with boundary conditions see [3, 7–9, 11].

We describe briefly the consequences for SB systems. The results are clear, but the second is long-winded to write down exactly.

With the construction method of Section 1 and its analogous version for minimal upper separable representations, we are able to construct a minimal SB realization for an integral operator $T: L_2([a, b], Z) \to L_2([a, b], Y)$, which is defined as

$$(T\varphi)(t) = D(t)\,\varphi(t) + \int_a^b k(t, s)\,\varphi(s)\,ds.$$

The assumptions on D, φ, and k are as above. Let us construct two pairs $\{F_l, G_l\}$, respectively $\{F_u, G_u\}$, with internal spaces X_l, respectively X_u, which are a minimal lower, respectively upper, separable representation of k. As in Section I.4 of [4], we construct the following SB realization of T:

$$\theta = \left[0, \begin{bmatrix} G_l(t) \\ G_u(t) \end{bmatrix}, [F_l(t) - F_u(t)], D(t), \begin{bmatrix} 0 & 0 \\ 0 & I_{X_u} \end{bmatrix}\right]_a^b, \tag{4.1}$$

where $X_l \oplus X_u$ is the state space of θ. The system θ is SB minimal because of Theorem 7.1 of [6].

Any SB system is always similar to one as in (4.1) (cf. [5, Sect. I.1; 6, Sect. 7]). So, we can describe the connection between two minimal SB systems with the help of Theorem 2.2 and its upper separable representation version. One can also derive necessary and sufficient conditions for an SB system to be minimal with Proposition 2.6 and its upper separable representation version.

It is easy to check that, if we use similar, respectively non-similar, minimal lower or upper separable representations of k to construct SB systems as in (4.1), then the resulting SB systems are similar, respectively non-similar. So, the number of non-similar minimal SB-realizations of T is either one or uncountable.

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